LINEAR DIOPHANTINE EQUATIONS

Equation of the form $ax + by = c$ where x and y are unknown integers.

Theorem: Let $a, b, c, \in \mathbb{Z}$, one of $a\ and\ b$ is not zero, then the ***Linear Diophantine Equation***

$ax + by = c$ has no solution if only if $(a, b)|c$.

Example: $a = 2,\quad b = 4,\quad c = 18$

Solution:

$$(2,4) \rightarrow \quad 2 = 1.2$$
$$4 = 2.2$$

So, $(2,4) = 2$. Then $2|18 \rightarrow 18 = 2\ (9)$.

Therefore, the Linear Diophantine Equation $\mathbf{2x + 4y = 18}$ has at least one solution.

- For $a = 3,\ b = 6,\ c = 17$, we have ***Linear Diophantine Equation*** $3x + 6y = 17$.

To check if it has a solution,
we have:

$$(3,6) \rightarrow \quad 3 = 1.3$$
$$6 = 2.3.1$$

So, $(3,6) = 3$. Then $3 \nmid 17$, for $17 = 2\ (8) + 1$

Thus, $\mathbf{3x + 6y = 17}$ has no solution.

Theorem: Let $d = (a, b)$. Let $X_{0,}Y_0$ be one particular solution to the Diophantine Equation $ax + by = c$. Then any other solution to the equation is given by:

$X = X_0 + (\frac{b}{d})t \qquad Y = Y_0 + (\frac{a}{d})t$

For all $t \in \mathbb{Z}$

Example: Find all positive integers x, y such that $4x + 6y = 100$.

Solution:

Using ***Euclidean Algorithm***, we solve for $(4,6)$.

$$6 = 4\,(1) + 2$$

$$4 = 2\,(2) + 0$$

Thus, $\boldsymbol{(4,6) = 2}$

By ***Bezout's Theorem***:

$$\Longrightarrow \quad 2 = 4x + 6y$$

$$\Longrightarrow \quad 2 = 4\,(-1) + 6\,(1)$$

$$\Longrightarrow \quad 2 = 2$$

Thus, $\boldsymbol{x = -1}$

$\boldsymbol{y = 1.}$

Then, $50\ [2 = 4(-1) + 6(1)]$

$$\Longrightarrow \quad 50\,(2) = 50\,(4.1) + 50\,(6.1)$$

$$\Longrightarrow \quad 100 = (50.4) - 1 + (50.6)\,1$$

$$\Longrightarrow \quad 100 = (4.50) - 1 \ + \ (6.50)\,1$$

$$\Longrightarrow \quad 100 \ = \ 4\,(50.-1) + \ 6\,(50.1)$$

$$\Longrightarrow \quad 100 \ = \ 4\,(-50) + \ \ 6\,(50)$$

Therefore, $\boldsymbol{X_0 = -50}$

$\boldsymbol{Y_0 = 50}$

Now using the formula from previous theorem,

we have:

$$\Rightarrow \ X = X_0 + \left(\frac{b}{d}\right)t \qquad\qquad Y = Y_0 + \left(\frac{a}{d}\right)t$$

$$\Rightarrow \ X = 50 + \left(\frac{6}{2}\right)t \qquad\qquad Y = 50 + \left(\frac{4}{2}\right)t$$

$$\Rightarrow \ X = -50 + 3t > 0 \qquad\qquad Y = -50 + 2t > 0$$

Solving for the possible values of $t \in \mathbb{Z}$,

we have:

$\Rightarrow$ $x = -50 + 3t > 0 \rightarrow -50 + 3t > 0 \rightarrow 3t > 50 \rightarrow t > \frac{50}{3}$ or $16\frac{2}{3}$.

$\Rightarrow$ $y = -50 + 2t > 0 \rightarrow -50 + 2t > 0 \rightarrow 50 > 2t \rightarrow 25 > t.$

$$\left.\begin{aligned} \textbf{\textit{Thus}}\ ,\ \boldsymbol{t \geq 17} \\ \boldsymbol{t < 25} \end{aligned}\right\} \boxed{\boldsymbol{17 \leq t < 25.}}$$

So, we have:

$\boldsymbol{t =}$	17	18	19	20	21	22	23	24
$\boldsymbol{x =}$	1	4	7	10	13	16	19	22
$\boldsymbol{y =}$	16	14	12	10	8	6	4	2

Thus, the solution are :

$\boldsymbol{x =}$	1
$\boldsymbol{y =}$	16

$\boldsymbol{x =}$	4
$\boldsymbol{y =}$	14

$\boldsymbol{x =}$	22
$\boldsymbol{y =}$	2

$\boldsymbol{x =}$	7
$\boldsymbol{y =}$	12

$\boldsymbol{x =}$	16
$\boldsymbol{y =}$	6

$\boldsymbol{x =}$	19
$\boldsymbol{y =}$	4

$\boldsymbol{x =}$	10
$\boldsymbol{y =}$	10

$\boldsymbol{x =}$	13
$\boldsymbol{y =}$	8

To verify if the values of x and y satisfy the given ***Linear Diophantine Equation,*** we have:

i. When $x = 1$ and $y = 16$

- $4x + 6y = 100$

$\Rightarrow$ $4\ (1) + 6\ (16) = 100$ ***Substituting the values of x and y***

$\Rightarrow$ $4 + 96 = 100$ ***Simplifying $\boldsymbol{4\ (1)}$ and 6 (16)***

$\Rightarrow\ 100 = 100$ *Simplifying* $\mathbf{4 + 96}$

Therefore, $\boldsymbol{x = 1}$ ***and*** $\boldsymbol{y = 16}$ ***is a solution to the Linear Diophantine Equation*** $\boldsymbol{4x + 6y = 100}$

ii. When $x = 4$ and $y = 14$

- $4x + 6y = 100$

$\Rightarrow\ 4\,(4) + 6\,(14) = 100$ ***Substituting the values of*** $\boldsymbol{x}$ ***and*** $\boldsymbol{y}$

$\Rightarrow\ 16 + 84 = 100$ ***Simplifying*** $\mathbf{4\,(4)}$ ***and*** $\mathbf{6\,(14)}$

$\Rightarrow\ 100 = 100$ ***Simplifying*** $\mathbf{16 + 84}$

Therefore, $\boldsymbol{x = 4}$ ***and*** $\boldsymbol{y = 14}$ ***is a solution to the Linear Diophantine Equation*** $\boldsymbol{4x + 6y = 100}$

iii. When $x = 7$ and $y = 12$

- $4x + 6y = 100$

$\Rightarrow\ 4\,(7) + 6\,(12) = 100$ ***Substituting the values of*** $\boldsymbol{x}$ ***and*** $\boldsymbol{y}$

$\Rightarrow\ 28 + 72 = 100$ ***Simplifying*** $\mathbf{4\,(7)}$ ***and*** $\mathbf{6\,(12)}$

$\Rightarrow\ 100 = 100$ ***Simplifying*** $\mathbf{28 + 72}$

Therefore, $\boldsymbol{x = 7}$ ***and*** $\boldsymbol{y = 12}$ ***is a solution to the Linear Diophantine Equation*** $\boldsymbol{4x + 6y = 100}$

iv. When $x = 10$ and $y = 10$

- $4x + 6y = 100$

$\Rightarrow\ 4\,(10) + 6\,(10) = 100$ ***Substituting the values of*** $\boldsymbol{x}$ ***and*** $\boldsymbol{y}$

$\Rightarrow\ 40 + 60 = 100$ ***Simplifying*** $\mathbf{4\,(10)}$ ***and*** $\mathbf{6\,(10)}$

$\Rightarrow\ 100 = 100$ ***Simplifying*** $\mathbf{40 + 60}$

Therefore, $\boldsymbol{x = 10}$ ***and*** $\boldsymbol{y = 10}$ ***is a solution to the Linear Diophantine Equation*** $\boldsymbol{4x + 6y = 100}$

v. When $x = 13$ and $y = 8$

- $4x + 6y = 100$

$\Rightarrow 4\,(13) + 6\,(8) = 100$ *Substituting the values of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$

$\Rightarrow 52 + 48 = 100$ *Simplifying* $\mathbf{4\,(13)}$ *and* $\mathbf{6\,(8)}$

$\Rightarrow 100 = 100$ *Simplifying* $\mathbf{52 + 48}$

Therefore, $\boldsymbol{x = 13}$ ***and*** $\boldsymbol{y = 8}$ ***is a solution to the Linear Diophantine Equation*** $\boldsymbol{4x + 6y = 100}$

vi. When $x = 16$ and $y = 6$

- $4x + 6y = 100$

$\Rightarrow 4\,(16) + 6\,(6) = 100$ *Substituting the values of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$

$\Rightarrow 64 + 36 = 100$ *Simplifying* $\mathbf{4\,(16)}$ *and* $\mathbf{6\,(6)}$

$\Rightarrow 100 = 100$ *Simplifying* $\mathbf{64 + 36}$

Therefore, $\boldsymbol{x = 16}$ ***and*** $\boldsymbol{y = 6}$ ***is a solution to the Linear Diophantine Equation*** $\boldsymbol{4x + 6y = 100}$

vii. When $x = 19$ and $y = 4$

- $4x + 6y = 100$

$\Rightarrow 4\,(19) + 6\,(4) = 100$ *Substituting the values of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$

$\Rightarrow 76 + 24 = 100$ *Simplifying* $\mathbf{4\,(19)}$ *and* $\mathbf{6\,(4)}$

$\Rightarrow 100 = 100$ *Simplifying* $\mathbf{52 + 48}$

Therefore, $\boldsymbol{x = 19}$ ***and*** $\boldsymbol{y = 4}$ ***is a solution to the Linear Diophantine Equation*** $\boldsymbol{4x + 6y = 100}$

viii. When $x = 22$ and $y = 2$

- $4x + 6y = 100$

$\Rightarrow 4\,(22) + 6\,(2) = 100$ *Substituting the values of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$

$\Rightarrow 88 + 12 = 100$ *Simplifying* $\mathbf{4\,(22)}$ *and* $\mathbf{6\,(2)}$

$\Rightarrow 100 = 100$ *Simplifying* $\mathbf{88 + 12}$

Therefore, $\boldsymbol{x = 22}$ ***and*** $\boldsymbol{y = 2}$ ***is a solution to the Linear Diophantine Equation*** $\boldsymbol{4x + 6y = 100}$